For the following questions, if the answer is true prove it, if the answer is false give a counter example. I encourage you all to atleast attempt them on your own first before looking at the solutions.

**Definition.** Given a natural number $n$ we define $n$ **factorial** or $n!$ as,

$$0! = 1,$$
$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n.$$

1. Determine if the following series converge or diverge.
   (a) $\sum_{n \geq 1} \frac{2n\sqrt{n^2+3}}{\sqrt{5n+2n^5}}$
   (b) $\sum_{n \geq 2} \frac{n!+1}{(n+1)!}$
   (c) $\sum_{n \geq 3} \frac{(2n)!}{(n!)^2}$
   (d) $\sum_{n \geq 2015} \frac{\sin^2(2015 \log n)}{n^{2015}}$
2. True and false:
   (a) If $a_n \geq 0$ and $\sum_n a_n$ converges then $\sum_n a_n^2$ converges.
   (b) If $a_n \geq 0$ and $\sum_n a_n^2$ converges then $\sum_n a_n$ converges.
3. Prove that if $|r| < 1$ then
$$\sum_{n=k}^{\infty} ar^n = \frac{ar^k}{1-r}.$$
4. Write $2.15\overline{62} (= 2.156262626262\ldots)$ as a fraction.

**Solution.**

1. (a) When $n$ is large we have

$$a_n \equiv \frac{2n\sqrt{n^2+3}}{\sqrt{5n+2n^5}} \approx \frac{2n\sqrt{n^2}}{\sqrt{2n^5}} = \frac{\sqrt{2}}{\sqrt{n}}.$$

So let $b_n = \frac{1}{\sqrt{n}}$, and note that,

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{2n\sqrt{n^2+3}}{\sqrt{5n+2n^5}}}{\frac{1}{\sqrt{n}}} \\
&= \lim_{n\to\infty} \frac{2n^{3/2}\sqrt{n^2+3}}{\sqrt{5n+2n^5}} \\
&= \lim_{n\to\infty} \frac{2n^{3/2}\sqrt{n^2+3}}{\sqrt{5n+2n^5}} \frac{\frac{1}{n^{5/2}}}{\frac{1}{n^{5/2}}} \\
&= \lim_{n\to\infty} \frac{2\frac{\sqrt{n^2+3}}{n}}{\frac{\sqrt{5n+2n^5}}{n^{5/2}}} \\
&= \lim_{n\to\infty} \frac{2\sqrt{1+\frac{3}{n^2}}}{\sqrt{\frac{5}{n^4}+2}} \\
&= \frac{2\sqrt{1+0}}{\sqrt{0+2}} \\
&= \sqrt{2} \\
&> 0.
\end{aligned}$$

We also have that $\sum_n b_n$ diverges by the $p$-test ($p = \frac{1}{2}$). So by limit comparison test we have $\sum_n a_n$ diverges.

(b) Note that

$$a_n \equiv \frac{n!+1}{(n+1)!} \geq \frac{n!}{(n+1)!} = \frac{1}{n+1}.$$

Since $\sum_n \frac{1}{n+1}$ diverges by the $p$-test ($p = 1$), we have by the comparison test $\sum_n a_n$ diverges.

(c) Let us first write out $\frac{(2n)!}{(n!)^2}$ in a neater way.

$$\begin{aligned}
\frac{(2n)!}{(n!)^2} &= \frac{1\cdot 2\cdots(n-1)\cdot n\cdot(n+1)\cdots(2n-2)\cdot(2n-1)\cdot(2n)}{1\cdot 2\cdots(n-1)\cdot n\cdot 1\cdot 2\cdots(n-1)\cdot n} \\
&= \frac{1}{1}\cdot\frac{2}{2}\cdots\frac{n-1}{n-1}\cdot\frac{n}{n}\cdot\frac{n+1}{1}\cdot\frac{n+2}{2}\cdots\frac{2n-1}{n-1}\cdot\frac{2n}{n} \\
&= \frac{n+1}{1}\cdot\frac{n+2}{2}\cdots\frac{2n-1}{n-1}\cdot\frac{2n}{n} \\
&\geq 1\cdot 1\cdots 1\cdot 1 \\
&= 1
\end{aligned}$$

In particular we have,

$$\lim_{n\to\infty} \frac{(2n)!}{(n!)^2} \neq 0.$$

So by the divergence test, $\sum_n \frac{(2n)!}{(n!)^2}$ diverges.

(d) We have that,

$$\frac{\sin^2(2015\log n)}{n^{2015}} \leq \frac{1}{n^{2015}}.$$

Since $\sum_n \frac{1}{n^{2015}}$ converges by $p$-test ($p = 2015$), we have that our series converges by the comparison test.

2. (a) **TRUE**

Since $\sum_n a_n$ converges we have that $\lim_{n\to\infty} a_n = 0$. So in particular we have that eventually $|a_n| < 1$, ie, there is an $N$ large enough such that when $n > N$ then $|a_n| < 1$. Since $x^2 < x$ when $|x| < 1$, we have when $n > N$,

$$a_n^2 \leq |a_n|.$$

So by comparison test, $\sum_n a_n^2$ converges.

(b) **FALSE**

Let $a_n = \frac{1}{n}$. We have that $\sum_n a_n^2 = \sum_n \frac{1}{n^2}$ converges by $p$-test ($p = 2$) but $\sum_n a_n = \sum_n \frac{1}{n}$ diverges by $p$-test ($p = 1$).

3. Let us begin by rewriting the series,

$$\begin{aligned}\sum_{n=k}^{\infty} ar^n &= ar^k + ar^{k+1} + ar^{k+2} + \cdots \\ &= ar^k(1 + r + r^2 \cdots) \\ &= ar^k \frac{1}{1-r}.\end{aligned}$$

The last line we used the fact that $(1 + r + r^2 + \cdots)$ is a geometric series and converges to $\frac{1}{1-r}$ when $|r| < 1$.

4. We can rewrite $2.15\overline{62}$ as follows,

$$\begin{aligned}2.15\overline{62} &= 2.15626262\ldots \\ &= 2.15 + 0.0062 + 0.000062 + \cdots \\ &= \frac{215}{100} + \frac{62}{10^4} + \frac{62}{10^6} \cdots \\ &= \frac{215}{100} + \frac{62}{10^4}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \cdots\right) \\ &= \frac{215}{100} + \frac{62}{10^4}\sum_{n=0}^{\infty}\frac{1}{100^n} \\ &= \frac{215}{100} + \frac{62}{10000}\frac{1}{1-\frac{1}{100}}.\end{aligned}$$

The last line is true since the series was geometric with ratio $1/100 < 1$. After simplifying the fraction we have,

$$2.15\overline{62} = \frac{21347}{9900}.$$